Classification of the invariant subspaces of the Cohen–Wales representation of the Artin group of type D_n

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Abstract. Recently, Cohen and Wales built a faithful linear representation of the Artin group of type D_n, hence showing the linearity of this group. It was later discovered that this representation is reducible for some complex values of its two parameters. It was also shown that when the representation is reducible, the action on a proper invariant subspace is a Hecke algebra action of type D_n. The goal of this paper is to classify these proper invariant subspaces in terms of Specht modules indexed by double partitions of the integer n. This work is the continuation of [9].

1 Introduction

In [2], Cohen and Wales built representations of the Artin groups of types A, D and E. In type A, their representation is equivalent to the Lawrence–Krammer representation of the braid group, a famous representation that was used in [1] and independently in [7] to show the linearity of the braid group. In [9], we use knot theory to construct a representation $\nu^{(n)}$ of an algebra that contains the Artin group of type D_n and that depends on two parameters l and m. This algebra is defined by the authors in [3] and is a generalization of the Birman-Murakami-Wenzl algebra to type D_n. We thus call it the CGW algebra of type D_n. We show that as a representation of the Artin group of type D_n and up to some change of parameters, this representation is equivalent to the Cohen-Wales representation of the Artin group type D_n. We prove that the representation is generically irreducible, but that when the parameters are specified to some nonzero complex numbers, it becomes reducible. We give a reducibility criterion for this representation, thus obtaining a reducibility criterion for the original representation of Cohen and Wales. The parameters t and r of the Cohen-Wales representation are related to the parameters l and m of the algebra by $m = r - \frac{1}{r}$ and $l = \frac{1}{t\,r^3}$. The complex parameters t and r for

which the Cohen-Wales representation is reducible are given in the Main Theorem of [9]. It is also shown in [9] that when the representation is reducible the action on a proper invariant subspace is a Hecke algebra action. The r of [9] and of the current paper is the $\frac{1}{r}$ of the Cohen-Wales representation. As in [9], we denote by $\mathcal{H}(D_n)$ the Hecke algebra of type D_n with parameter r^2 over the field $\mathbb{Q}(l,r)$. The classes of irreducible $\mathcal{H}(D_n)$-modules are called Specht modules and are indexed by double partitions of the integer n, see § 3.2 of [9]. We denote by $\mathcal{H}_{F,r^2}(n)$ the Iwahori-Hecke agebra of the symmetric group $Sym(n)$ with parameter r^2 over the field $\mathbb{Q}(l,r)$. Some important aspects of the representation theory of $\mathcal{H}_{F,r^2}(n)$ can be found in [10] and some elements of the representation theory of $\mathcal{H}(D_n)$ appear for instance in [5] and in [4]. We call the representation $\nu^{(n)}$ introduced in [9] the Cohen-Wales representation of the CGW algebra of type D_n. We prove the following Theorem.

Theorem 1. *Let n be an integer with $n \geq 4$. Assume $\mathcal{H}(D_n)$ and $\mathcal{H}_{F,r^2}(n)$ are both semisimple. Thus, assume that $r^{2k} \notin \{-1,1\}$ for every integer k with $1 \leq k \leq n-1$ and $r^{2n} \neq 1$.*
1) *Suppose first $n \geq 5$. There are two cases.*

(i) Assume $r^{2(n-1)} \notin \{i,-i\}$ if $l=-r^3$. When the Cohen-Wales representation $\nu^{(n)}$ of the CGW algebra of type D_n of degree n^2-n is reducible, its unique proper invariant subspace is isomorphic to one of the Specht modules

$$S^{(0),(n)},\ \ S^{(0),(n-1,1)},\ \ S^{(1),(n-1)},\ \ S^{(0),(n-2,2)},\ \ S^{(2),(n-2)},\ \ S^{(1),(n-2,1)},$$

which respectively arise if and only if

$$l=\frac{1}{r^{4n-7}},\ \ l=\frac{1}{r^{2n-7}},\ \ l=-\frac{1}{r^{2n-5}},\ \ l=r^3,\ \ l=\frac{1}{r},\ \ l=-r^3$$

(ii) If $l=-r^3=\frac{1}{r^{4n-7}}$, there are exactly three proper invariant subspaces in the Cohen-Wales space, and they are respectively isomorphic to

$$S^{(0),(n)},\ S^{(1),(n-2,1)}\ \text{and}\ \ S^{(0),(n)}\oplus S^{(1),(n-2,1)}$$

2) *Case $n=4$. There are three cases.*

(i) Assume that $r^6 \neq i$ and $r^6 \neq -i$ when $l=-r^3$. Assume also that $l \neq \frac{1}{r}$. When the Cohen-Wales representation $\nu^{(4)}$ of the CGW algebra of type D_4 of degree 12 is reducible, its unique proper invariant subspace is isomorphic to one of the Specht modules

$$S^{(0),(4)},\ S^{(0),(2,2)},\ S^{(1),(3)},\ S^{(1),(2,1)},$$

which respectively arise if and only if

$$l=\frac{1}{r^9},\ l=r^3,\ l=-\frac{1}{r^3},\ l=-r^3$$

(ii) Same as (ii) in the general case 1).

(iii) if $l = \frac{1}{r}$, there are exactly seven proper invariant subspaces in the Cohen-Wales space and they are respectively isomorphic to

$$S^{(0),(3,1)},\ S^{(2,2)^+},\ S^{(2,2)^-},\ S^{(0),(3,1)} \oplus S^{(2,2)^+},\ S^{(0),(3,1)} \oplus S^{(2,2)^-},\ S^{(2,2)^+} \oplus S^{(2,2)^-},$$
$$S^{(0),(3,1)} \oplus S^{(2,2)^+} \oplus S^{(2,2)^-}$$

A summary of the Specht modules that occur in the Cohen-Wales space and the values of the parameters for which they occur is given in Table 1 below.

Note (ii) is the only case when two of the reducibility values for l can be equal in the case when $n \geq 5$. When $n = 4$, two of the reducibility values for l in the generic case are identical, which makes this case special and described in (iii).

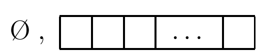
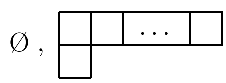
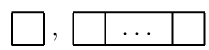
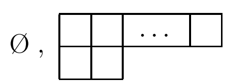
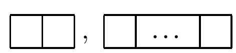
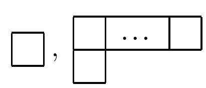
Reducibility value	Specht module	Double partition	Dimension
$l = \frac{1}{r^{4n-7}}$	$S^{(0),(n)}$		1
$l = \frac{1}{r^{2n-7}}$	$S^{(0),(n-1,1)}$		$n-1$
$l = -\frac{1}{r^{2n-5}}$	$S^{(1),(n-1)}$		n
$l = r^3$	$S^{(0),(n-2,2)}$		$\frac{n(n-3)}{2}$
$l = \frac{1}{r}$	$S^{(2),(n-2)}$		$\frac{n(n-1)}{2}$
$l = -r^3$	$S^{(1),(n-2,1)}$		$n(n-2)$

Table 1. Classification of the invariant subspaces of the Cohen-Wales representation of the CGW algebra of type D_n.

Our proof is based on the existing theorems of [9]. We do the three following things. We identify the dimensions of the irreducible $\mathcal{H}(D_n)$-modules that may occur in the Cohen-Wales space when $n \geq 11$, under some assumption at rank 10. We prove the classification theorem for these n, assuming the classification

theorem holds when $4 \leq n \leq 10$ and we use in particular some results of [8]. We deal with the small cases $4 \leq n \leq 10$.

2 Proof of the classification theorem

In [9], $\S\, 3.2$, we describe the degrees of the irreps of $\mathcal{H}(D_n)$ that are less than $n^2 - n$, the degree of the Cohen-Wales representation of type D_n. Our study combined with Theorem 7 of [6] shows that for $n \geq 9$, an irreducible $\mathcal{H}(D_n)$-module of dimension less than $n^2 - n$ is one of the Specht modules $S^{(0),(n)}$, $S^{(0),(n-1,1)}$, $S^{(0),(n-2,2)}$, $S^{(0),(n-2,1,1)}$, $S^{(1),(n-1)}$, $S^{(1),(n-2,1)}$, $S^{(2),(n-2)}$, $S^{(1,1),(n-2)}$, or one of their conjugates, or is a Specht module $S^{(0),\lambda}$ for some partition λ of n, whose dimension lies between $\frac{(n-1)(n-2)}{2}$ and $n^2 - n$. The Specht modules that were listed first are of respective dimensions 1, $n-1$, $\frac{n(n-3)}{2}$, $\frac{(n-1)(n-2)}{2}$, n, $n(n-2)$ and $\frac{n(n-1)}{2}$ for the last two. Our goal next is to show that the Specht modules of the second category cannot occur in the Cohen-Wales space V_n when $n \geq 11$, under some assumption at rank 10. We introduce some convenient notations.

Definition 1. *We will denote by $Q_n(m)$ the set of Specht modules $S^{(0),\lambda}$ with the first row or first column of the Ferrers diagram of the partition λ of n containing at least $n - m$ boxes.*

When $n \geq 11$, we have seen at the end of $\S\, 3.2$ of [9] that there are no elements of $Q_n(3) \setminus Q_n(2)$ occuring in V_n as their dimensions are too big. This is the key remark to further show the following proposition under hypothesis $(\mathcal{H})_{\mathbf{10}}$.

$(\mathcal{H})_{\mathbf{k}}$: **The Specht modules $S^{(0),\lambda}$ occurring in V_k belong to $Q_k(2)$.**

Proposition 1. $(\mathcal{H})_{\mathbf{10}} \Rightarrow \forall n \geq 11,\ (\mathcal{H})_{\mathbf{n}}$

PROOF. Let $n \geq 11$ and suppose the property holds at rank $n-1$. Let $\mathcal{W}$ be a proper invariant subspace of V_n such that $\mathcal{W}$ is isomorphic to $S^{(0),\lambda}$ for some partition λ of n. Because $n \geq 11$, we already know that $\mathcal{W} \notin Q_n(3) \backslash Q_n(2)$. Suppose $\mathcal{W} \in Q_n(m) \backslash Q_n(3)$ for some integer $m \geq 4$. Since in particular $\mathcal{W} \notin Q_n(2)$ and $n \geq 9$, Theorem 7 of [6] implies that $\dim(\mathcal{W}) > \frac{(n-1)(n-2)}{2}$. And so $\mathcal{W} \cap V_{n-1} \neq \{0\}$. Moreover, we see with the branching rule that $\mathcal{W} \downarrow_{\mathcal{H}(D_{n-1})}$ is isomorphic to a direct sum of an element of $Q_{n-1}(m-1) \setminus Q_{n-1}(2)$ and an element of $Q_{n-1}(m) \setminus Q_{n-1}(3)$. But by induction $\mathcal{W} \cap V_{n-1}$ belongs to $Q_{n-1}(2)$, hence a contradiction and the result.

The discussion at the beginning of $\S\, 2$ and Proposition 1 imply Proposition 2.

Proposition 2. *Assume $(\mathcal{H})_{10}$ holds. Then, for any $n \geq 11$, the Specht modules that may occur in the Cohen-Wales space V_n have dimensions*

$$1, n-1, n, \frac{n(n-3)}{2}, \frac{(n-1)(n-2)}{2}, \frac{n(n-1)}{2}, n(n-2)$$

The discussion at the beginning of § 2, Proposition 2 and the theorems of the introduction of [9] and their proofs imply in turn Proposition 3.

Proposition 3. *Assume Theorem* 1 *holds for integers* n *with* $4 \leq n \leq 10$. *Then, for any* $n \geq 4$, *the Specht modules that may occur in the Cohen-Wales space* V_n *are*

$$S^{(0),(n)}, S^{(0),(n-1,1)}, S^{(0),(n-2,2)}, S^{(0),(n-2,1,1)}, S^{(0),(3,1^{n-3})},$$
$$S^{(1),(n-1)}, S^{(1),(n-2,1)}, S^{(2),(n-2)}$$

PROOF. It suffices to show that the Specht modules $S^{(1,1),(n-2)}$, $S^{(1,1),(1^{n-2})}$, $S^{(2),(1^{n-2})}$ when $n \geq 5$, the Specht module $S^{(1,1),(2)}$ when $n = 4$, and the Specht module $S^{(1),(2,1^{n-3})}$ when $n \geq 5$ cannot occur in V_n.

First, we deal with the case $n = 4$. We show the following result. We use the notations of [9], where the vectors w_{ij}'s and $\widehat{w_{ij}}$'s for $1 \leq i < j \leq n$ denote the basis vectors of the Cohen-Wales space. For the precise definition of the Cohen-Wales representation of the CGW algebra of type D_n, we refer the reader to Theorem 1 of [9].

Result 1. (i) *There exists in* V_4 *an irreducible* 3*-dimensional invariant subspace isomorphic to* $S^{(2,2)^+}$ *(resp* $S^{(2,2)^-}$*) if and only if* $l = \frac{1}{r}$. *If so, it is unique.*
Moreover, the 6*-dimensional invariant subspace spanned by the vectors* $t_{ij} = w_{ij} - \widehat{w_{ij}}$ *with* $1 \leq i < j \leq 4$ *decomposes as a direct sum of two irreducible invariant subspaces* $\mathcal{U}$ *and* $\mathcal{W}$, *both of dimension* 3 *and respectively spanned by the vectors*

$$\left\{\begin{array}{rcl} u_1 &=& m(w_{12} - \widehat{w_{12}}) + (w_{13} - \widehat{w_{13}}) + (w_{24} - \widehat{w_{24}}) \\ u_2 &=& (1-r^2)(w_{12} - \widehat{w_{12}}) + (w_{23} - \widehat{w_{23}}) + (w_{14} - \widehat{w_{14}}) - m\,(w_{24} - \widehat{w_{24}}) \\ u_3 &=& (w_{12} - \widehat{w_{12}}) + (w_{34} - \widehat{w_{34}}) \end{array}\right.$$

$$\left\{\begin{array}{rcl} w_1 &=& -(w_{13} - \widehat{w_{13}}) + m\,(w_{23} - \widehat{w_{23}}) + (w_{24} - \widehat{w_{24}}) \\ w_2 &=& -(w_{12} - \widehat{w_{12}}) + m\,(w_{13} - \widehat{w_{13}}) + (1-r^2)(w_{23} - \widehat{w_{23}}) + (w_{34} - \widehat{w_{34}}) \\ w_3 &=& -(w_{23} - \widehat{w_{23}}) + (w_{14} - \widehat{w_{14}}) \end{array}\right.$$

(ii) *The Specht modules* $S^{(1^2,1^2)^+}$ *and* $S^{(1^2,1^2)^-}$ *don't occur in* V_4.
(iii) *The Specht module* $S^{(1,1),(2)}$ *does not occur in* V_4.

PROOF. Assume $l = \frac{1}{r}$. Then, for any $n \geq 4$, the t_{ij}'s span an $\frac{n(n-1)}{2}$-dimensional invariant subspace of V_n. When $n \geq 5$, it is shown in § 4 of [9] that this invariant subspace of V_n is irreducible. We show when $n = 4$, the 6-dimensional invariant subspace $\mathcal{T}$ spanned by the t_{ij}'s is reducible and decomposes as a direct sum of two irreducible 3-dimensional invariant subspaces of V_4. We formed the matrices G_i's of the left actions by the g_i's, $1 \leq i \leq 4$ in the basis $(w_{ij} - \widehat{w_{ij}})_{1 \leq i < j \leq 4}$ of $\mathcal{T}$. Using Maple, we then computed the commutant of these matrices and derived the following informations. The commutant has dimension 2 and is spanned by a homothety and another matrix A. The latter matrix is symmetric and has two eigenvalues, namely

$$\lambda_1 = \frac{1 - r^2 + 2r^4}{r(1-r^2)} \qquad \text{and} \qquad \lambda_2 = \frac{r(r^4 - r^2 + 2)}{r^2 - 1}$$

The corresponding eigenspaces both have dimension 3 and are respectively spanned by the vectors u_1, u_2, u_3 and w_1, w_2, w_3 of $\mathcal{T}$ defined in the statement of the result. Since A commutes to the G_i's, if v_k is an eigenvector of A for the eigenvalue λ_k, then $G_i v_k$ is also an eigenvector of A for the same eigenvalue. So, the vector spaces $\mathcal{U}$ and $\mathcal{W}$ are invariant under the actions by the g_i's. We provide below the matrices M_i's (resp N_i's) of the left actions by the g_i's in the basis (u_1, u_2, u_3) (resp (w_1, w_2, w_3)) of $\mathcal{U}$ (resp $\mathcal{W}$).

$$M_2 = \begin{bmatrix} 0 & 1 & 0 \\ 1 & -m & 0 \\ 0 & 0 & r \end{bmatrix}$$

$$M3 = \begin{bmatrix} 0 & m\,r & 1 \\ 0 & r & 0 \\ 1 & -m & -m \end{bmatrix} \qquad M1 = \begin{bmatrix} -m & -1 & \frac{m}{r} \\ -1 & 0 & m \\ 0 & 0 & r \end{bmatrix}$$

$$M4 = \begin{bmatrix} 0 & 1 & 0 \\ 1 & -m & 0 \\ 0 & 0 & r \end{bmatrix}$$

Matrix representation for the $\mathcal{H}(D_4)$-module $\mathcal{U}$.

$$N_2 = \begin{bmatrix} -m & 0 & 1 \\ 0 & r & 0 \\ 1 & 0 & 0 \end{bmatrix}$$

$$N_3 = \begin{bmatrix} 0 & 1 & 0 \\ 1 & -m & 0 \\ 0 & 0 & r \end{bmatrix} \qquad N_1 = \begin{bmatrix} 0 & m\,r & -1 \\ 0 & r & 0 \\ -1 & m & -m \end{bmatrix}$$

$$N_4 = \begin{bmatrix} 0 & m\,r & -1 \\ 0 & r & 0 \\ -1 & m & -m \end{bmatrix}$$

Matrix representation for the $\mathcal{H}(D_4)$-module $\mathcal{W}$.

To show that $\mathcal{U}$ and $\mathcal{W}$ are irreducible, it suffices to notice that the existence of a one-dimensional invariant subspace inside them would force $l = \frac{1}{r^9}$ by Theorem 3 of [9]. This not compatible with $l = \frac{1}{r}$ and $r^8 \neq 1$.
When $l = \frac{1}{r}$, there exists in V_4 a **unique** irreducible 3-dimensional invariant subspace $\mathcal{V}$ that is isomorphic to $S^{(0),(3,1)}$ by the proof of Theorem 4 of [9] and there does not exist any irreducible 3-dimensional invariant subspace of V_4 that is isomorphic to $S^{(0),(2,1,1)}$. Since $\mathcal{U} \neq \mathcal{V}$ and $\mathcal{W} \neq \mathcal{V}$, the two irreducible matrix representations of $\mathcal{H}(D_4)$ of degree 3 defined by the matrices M_i's and N_i's above are not equivalent to any of those defined by the relations (Δ) and (∇) on pages 24 and 25 of [9]. Then they must be matrix representations for $S^{(2,2)^+}$ or $S^{(2,2)^-}$. Moreover, using for instance Maple, we see that the two

matrix representations $(M_i)_{1\leq i\leq 4}$ and $(N_i)_{1\leq i\leq 4}$ are inequivalent. Then, the sufficient conditions in point (i) of Result 1 hold.
To finish the proof of Result 1, we prove the following Proposition.

Proposition 4.
(i) If there exists in V_4 an irreducible 3-dimensional invariant subspace, then $l = \frac{1}{r}$.
(ii) If there exists in V_4 an irreducible 6-dimensional invariant subspace, then $l = \frac{1}{r}$.

First we show a lemma.

Lemma 1. *If in the Cohen-Wales space V_4 there exists an irreducible 2-dimensional invariant subspace isomorphic to $S^{(0),(2,2)}$, then $l = r^3$.*

A consequence of that lemma is that Theorem 6 of [9] holds in the case $n = 4$ as well. To show the lemma, we note that the Specht module $S^{(0),(2,2)}$ is the only irreducible $\mathcal{H}(D_4)$-module of dimension 2 and we use the irreducible matrix representation of degree 2 of $\mathcal{H}(D_4)$ that was given on page 32 of [9]. Suppose there exists two linearly independent vectors u_1 and u_2 of V_4 such that

$$\forall\, i \in \{1,2,4\},\ \left\{\begin{array}{lcl} g_i\,u_1 & = & -\frac{1}{r}\,u_1 \quad \spadesuit_i \\ g_i\,u_2 & = & u_1 + r\,u_2 \end{array}\right. \quad \text{and} \quad \left\{\begin{array}{lcl} g_3\,u_1 & = & r\,u_1 + u_2 \\ g_3\,u_2 & = & -\frac{1}{r}\,u_2 \end{array}\right.$$

Then, by $\spadesuit_2$ and $\spadesuit_4$,

$$u_1 = \overline{\lambda_{13}}\,\overline{w_{13}} - \frac{1}{r}\,\overline{\lambda_{13}}\overline{w_{14}} + \frac{1}{r^2}\,\overline{\lambda_{13}}\overline{w_{24}} - \frac{1}{r}\,\overline{\lambda_{13}}\overline{w_{23}} \tag{1}$$

Moreover, by $\spadesuit_1$, we have $\lambda_{13} = \frac{1}{r^2}\,\widehat{\lambda_{13}}$. In particular, all the coefficients in (1) are non-zero. Without loss of generality, take $\lambda_{13} = 1$. So, a complete expression for u_1 is

$$u_1 = w_{13} - \frac{1}{r}\,w_{14} + \frac{1}{r^2}\,w_{24} - \frac{1}{r}\,w_{23} + r^2\big(\widehat{w_{13}} - \frac{1}{r}\,\widehat{w_{14}} + \frac{1}{r^2}\,\widehat{w_{24}} - \frac{1}{r}\,\widehat{w_{23}}\big) \tag{2}$$

Now, the coefficient in $\widehat{w_{12}}$ in $u_2 = g_3.u_1 - r\,u_1$ is r^2. By looking at the coefficient of $\widehat{w_{12}}$ in $g_1\,u_2 = u_1 + r\,u_2$, we hence derive

$$\frac{r^4}{l} - \frac{m}{r^2} + m^2\left(\frac{1}{r} + r\right) = r^3$$

After simplification, this yields $l = r^3$.
In [9], we introduce a $\mathcal{CGW}(D_n)$-submodule $K(n)$ of V_n which has the nice property that it contains all the proper invariant subspaces of V_n when the representation $\nu^{(n)}$ is reducible. We denoted its dimension by $k(n)$ (see Definition 3 of § 3.6.1 of [9]). In [9], we use a program in Mathematica that computes a matrix whose kernel is $K(n)$. Our program runs until rank 7 and computes $k(n)$ for the different values of l and r. By Theorem 2 point (i) of [9], we know that if $\nu^{(4)}$ is reducible, then

$$l \in \left\{\frac{1}{r^9}, -r^3, r^3, -\frac{1}{r^3}, \frac{1}{r}\right\}$$

The corresponding values for $k(4)$ are in the same order

$$k(4) \in \{1, 8, 2, 4, 9\},$$

except when $l = \frac{1}{r^9} = -r^3$ when $k(4) = 9$. Suppose there exists in V_4 an irreducible 3-dimensional invariant subspace. Then $k(4) \geq 3$, and so $l \neq r^3$. Also, we have $l \neq -\frac{1}{r^3}$. Indeed, otherwise the irreducible 3-dimensional invariant subspace would have a one-dimensional $\mathcal{H}(D_4)$-summand in $K(4)$. This would force $l = \frac{1}{r^9}$ by Theorem 3 of [9]. But we have $r^6 \neq -1$ with our restrictions on r. Suppose $l = -r^3$ and $r^{12} \neq -1$. Then we have $k(4) = 8$. Then, the irreducible 3-dimensional invariant subspace has a summand in $K(4)$ of dimension 5. By the representation theory of $\mathcal{H}(D_4)$, this summand is not irreducible. By Lemma 1, this summand cannot contain an irreducible 2-dimensional invariant subspace. Nor can it contain an irreducible 4-dimensional invariant subspace by Theorem 5, point (i) of [9]. So, it is impossible to have $k(4) = 8$. We cannot have $l = -r^3 = \frac{1}{r^9}$ either. Indeed, if $k(4) = 9$, the irreducible 3-dimensional invariant subspace has a summand in $K(4)$ of dimension 6. Moreover, since $l = \frac{1}{r^9}$, this summand must contain a one-dimensional invariant subspace. Then there exists in V_4 a 5-dimensional invariant subspace and we conclude as before, using also the uniqueness part in Theorem 3 of [9]. The only possibility that is left is hence to have $l = \frac{1}{r}$. This finishes the proof of (i). Suppose there exists in V_4 an irreducible 6-dimensional invariant subspace. Name it $\mathcal{V}_6$. Then $k(4) \geq 6$ and so $k(4) \in \{8, 9\}$. If $k(4) = 9$ and $l = \frac{1}{r}$, then we are done. Suppose $k(4) = 9$ and $l = \frac{1}{r^9} = -r^3$. Then $\mathcal{V}_6$ has a 3-dimensional summand in $K(4)$. Since $l \neq r^3$, this summand must be irreducible. Then an application of point (i) yields $l = \frac{1}{r}$, a contradiction. Finally, if $k(4) = 8$, then $\mathcal{V}_6$ has a 2-dimensional summand in $K(4)$. By uniqueness of a one-dimensional invariant subspace of V_4 when it exists, this summand must be irreducible. It then follows from Lemma 1 that $l = r^3$. This is impossible.

Using Proposition 4, we close the proof of Result 1 as follows. For (ii), we use the fact that when $\nu^{(4)}$ is reducible, it is indecomposable (this is a consequence of Proposition 1 of § 3.1 of [9]). The same argument yields the uniqueness part in point (i).
For (iii), we use the fact that if $S^{(1,1),(2)}$ occurs in V_4, then by Proposition 4 point (ii), we get $l = \frac{1}{r}$. Then, the sum of the degrees of the irreducible $\mathcal{H}(D_4)$-modules occurring in V_4 exceeds the degree of the Cohen-Wales representation of the CGW algebra of type D_4.

We state below a corollary of Result 1 and of the proof of Theorem 4 of [9].

Corollary 1. *There exists in V_4 an irreducible 3-dimensional invariant subspace if and only if $l = \frac{1}{r}$. If so, V_4 contains exactly three irreducible invariant subspaces of dimension 3. They are respectively spanned by the u_i's and the w_i's of Result 1 and the v_i's that were defined in Theorem 4 of [9].*

We now continue the proof of Proposition 3 by showing the following result.

Result 2. *For any $n \geq 5$, the Specht modules $S^{(2),(1^{n-2})}$, $S^{(1,1),(1^{n-2})}$ and $S^{(1),(2,1^{n-3})}$ cannot occur in V_n.*

Indeed, let $n \geq 5$ and suppose $\mathcal{W} \subset V_n$ and $\mathcal{W} \simeq S^{(2),(1^{n-2})}$. Then

$$\mathcal{W} \downarrow_{\mathcal{H}(D_{n-1})} \simeq S^{(1),(1^{n-2})} \oplus S^{(2),(1^{n-3})}$$

But by the key arguments at the end of § 3.5 of [9], the Specht module $S^{(1),(1^{n-2})}$ cannot be a constituent of $\mathcal{W} \downarrow_{\mathcal{H}(D_{n-1})}$.
The argument is identical for $S^{(1,1),(1^{n-2})}$ and for $S^{(1),(2,1^{n-3})}$, hence the result.

We further show the following Theorem, which gives a necessary condition on l and r for the existence of an irreducible $\frac{n(n-1)}{2}$-dimensional invariant subspace of V_n when $n \geq 5$, under some conditions when $n = 7$ or $n \geq 9$.

Theorem 2. *Let n be an integer with $n \geq 5$. When $n = 7$ or $n \geq 9$, assume that $(\mathcal{H})_n$ holds. If in the Cohen-Wales space V_n there exists an irreducible $\frac{n(n-1)}{2}$-dimensional invariant subspace, then $l = \frac{1}{r}$. Moreover, it is isomorphic to the Specht module $S^{(2),(n-2)}$.*

PROOF. The hypothesis ensures that there are no irreducible Specht modules $S^{(0),\lambda}$ with λ a partition of n of dimension $\frac{n(n-1)}{2}$ occurring in V_n when $n \geq 9$. As for $n = 7$, it will become clear later why we don't need to study the occurrence inside V_7 of $S^{(0),(3,3,1)}$ or its conjugate Specht module, both of dimension 21.

First, we deal with some special cases. We show that the Specht module $S^{(0),(5,3)}$ of dimension 28 cannot occur in V_8. If $\mathcal{W}$ is an invariant subspace of V_8 that is isomorphic to $S^{(0),(5,3)}$, then

$$\mathcal{W} \downarrow_{\mathcal{H}(D_5)} \simeq 3\, S^{(0),(3,2)} \oplus 3\, S^{(0),(4,1)} \oplus S^{(0),(5)}$$

But the proof of Lemma 3 point (i) of [9] shows that if there exists an invariant subspace of $\mathcal{W} \downarrow_{\mathcal{H}(D_5)}$ that is isomorphic to $S^{(0),(3,2)}$, then it is unique. Hence a contradiction.

We now eliminate in turn the conjugate Specht module $S^{(0),(2^3,1^2)}$. If $\mathcal{W}$ is an invariant subspace of V_8 isomorphic to $S^{(0),(2^3,1^2)}$, we have

$$\mathcal{W} \downarrow_{\mathcal{H}(D_6)} \simeq 2\, S^{(0),(2^2,1^2)} \oplus S^{(0),(2^3)} \oplus S^{(0),(2,1^4)}$$

Moreover, since $28 > 26$, we have $\mathcal{W} \cap V_6 \neq \{0\}$. By Lemma 3 point (ii) of [9] (resp the end of § 3.4 in [9] , resp the proof of Theorem 4 of [9]), the Specht module $S^{(0),(2^2,1^2)}$ (resp $S^{(0),(2^3)}$, resp $S^{(0),(2,1^4)}$) cannot occur in V_6. Hence we get a contradiction.
We deal with the last special case, namely $S^{(1),(2,2)}$. If $\mathcal{W}$ is an invariant subspace of V_5 that is isomorphic to $S^{(1),(2,2)}$, then

$$\mathcal{W} \downarrow_{\mathcal{H}(D_4)} \simeq S^{(0),(2,2)} \oplus S^{(1),(2,1)}.$$

The presence of a fifth node does not modify the proof of Lemma 1, and we derive $l = r^3$. Then, by using Mathematica, we have $k(5) = 5$. But the existence of an invariant subspace of V_5 of dimension 10 forces $k(5) \geq 10$, hence a contradiction.

We now deal with the general case. We show that $S^{(1,1),(n-2)}$ cannot occur in V_n.
Notice when $n \geq 5$, we have $\frac{n(n-1)}{2} > 2(n-1)$. So, if $\mathcal{W} \simeq S^{(1,1),(n-2)}$, then $\mathcal{W} \cap V_{n-1} \neq \{0\}$. First, we deal with the cases $n = 5$ and $n = 6$, then we proceed by induction on $n \geq 6$. Let $\mathcal{W} \subset V_5$ with $\mathcal{W} \simeq S^{(1,1),(3)}$. Then, $\mathcal{W} \downarrow_{\mathcal{H}(D_4)} \simeq S^{(1),(3)} \oplus S^{(1,1),(2)}$. On the one hand, by point (iii) of Result 1, we know that $\mathcal{W} \cap V_4$ cannot be isomorphic to $S^{(1,1),(2)}$. On the other hand, if $\mathcal{W} \cap V_4 \simeq S^{(1),(3)}$, then by Result 1 of § 3.5 of [9], we get $l = -\frac{1}{r^3}$. As both $\nu^{(4)}$ and $\nu^{(5)}$ are reducible, under our restrictions on r, we must then have $l = -\frac{1}{r^3} = \frac{1}{r^{13}}$. By using Mathematica as explained before, we computed $k(5) = 1$ when $l = \frac{1}{r^{13}}$ and $l \neq -r^3$. This is not compatible with $\dim(\mathcal{W}) = 10$ and $\mathcal{W} \subseteq K(5)$.
Suppose now $\mathcal{W}$ is an invariant subspace of V_6 with $\mathcal{W} \simeq S^{(1,1),(4)}$. And so $\mathcal{W} \downarrow_{\mathcal{H}(D_5)} \simeq S^{(1,1),(3)} \oplus S^{(1),(4)}$. Suppose $\mathcal{W} \cap V_5 \simeq S^{(1),(4)}$. Then, by using the same results as before, we get $l = -\frac{1}{r^5} = \frac{1}{r^{17}}$. Again, the contradiction comes from $k(6) = 1$ when $l = \frac{1}{r^{17}}$ and $l \neq -r^3$. Hence $\mathcal{W} \cap V_5$ is isomorphic to $S^{(1,1),(3)}$ instead. By the case $n = 5$, this is impossible.
Let $n \geq 7$ and suppose $S^{(1,1),(n-3)}$ does not occur in V_{n-1}. Let $\mathcal{W} \subset V_n$ with $\mathcal{W} \simeq S^{(1,1),(n-2)}$. Then,

$$\mathcal{W} \downarrow_{\mathcal{H}(D_{n-1})} \simeq S^{(1),(n-2)} \oplus S^{(1,1),(n-3)}$$

Since

$$\dim(\mathcal{W} \cap V_{n-1}) \geq \frac{n(n-1)}{2} - 2(n-1) = \frac{(n-1)(n-4)}{2},$$

we see that $\dim(\mathcal{W} \cap V_{n-1}) > n - 1$ as soon as $n \geq 7$. This contradicts the fact that $S^{(1,1),(n-3)}$ does not occur in V_{n-1}.
It remains to show that if $\mathcal{W}$ is an invariant subspace of V_n that is isomorphic to $S^{(2),(n-2)}$, then $l = \frac{1}{r}$. Again we proceed by induction on $n \geq 5$ and adapt the proof above. Suppose $\mathcal{W}$ is an invariant subspace of V_5 that is isomorphic to $S^{(2),(3)}$. By the same arguments as above, the Specht module $S^{(1),(3)}$ cannot occur in $\mathcal{W} \cap V_4$. Hence $S^{(2,2)^+}$ or $S^{(2,2)^-}$ or both Specht modules must occur in $\mathcal{W} \cap V_4$. Then, by Result 1 point (i), it follows that $l = \frac{1}{r}$. Suppose now $\mathcal{W}$ is an invariant subspace of V_6 that is isomorphic to $S^{(2),(4)}$. Then $\mathcal{W} \cap V_5$ must be isomorphic to $S^{(2),(3)}$. Then, by the case $n = 5$, we get $l = \frac{1}{r}$.
Finally, when $n \geq 7$, if $\mathcal{W}$ is an invariant subspace of V_n that is isomorphic to $S^{(2),(n-2)}$, by the same arguments as above, $S^{(2),(n-3)}$ occurs in V_{n-1} and by induction this forces $l = \frac{1}{r}$. This finishes the proof of the theorem.

We now better Proposition 2 and Proposition 3 by showing Result 3 and Result 4

Result 3. *Let $n \geq 4$. The Specht module $S^{(0),(n-2,1,1)}$ cannot occur in the Cohen-Wales space V_n.*

PROOF. When $n = 4$, this result is part of the proof of Theorem 4 of [9] in the case $n = 4$. Assume now $n \geq 5$. We will use the results of [8] to give a matrix representation for $S^{(0),(n-2,1,1)}$. In [8] § 2, we give a new representation of the braid group on n strands that is equivalent to the Lawrence–Krammer representation. In that paper, the vectors u_1, u_2, u_3 of Theorem 3.5 and the vectors $V_s^{(k)}$ with $5 \leq k \leq n$, $1 \leq s \leq k-2$ of Theorem 3.15 form a basis $\mathcal{B}$ of vectors of the Lawrence–Krammer space. It is a result of the paper that the matrices of the left braid group actions by the g_i's, $1 \leq i \leq n-1$, in this basis provide a matrix representation for the $\mathcal{H}_{F,r^2}(n)$-Specht module $S^{(n-2,1,1)}$. The left braid group actions by the g_i's on the basis of the Lawrence–Krammer space is the braid group representation provided in § 2 of [8]. The braid group actions by the g_i's on the family of vectors $(V_s^{(k)})_{5\leq k\leq n,\, 1\leq s\leq n-2}$ is the object of Lemma 3.16 of [8]. Moreover, we show along the proof of Theorem 3.15 in [8] that if $r^8 \neq -1$, another equivalent irreducible matrix representation for $S^{(n-2,1,1)}$ is provided by the matrices of the left braid group actions by the g_i's, $1 \leq i \leq n-1$ in the basis $\mathcal{B}'$ consisting this time of all the vectors $V_i^{(k)}$ with $3 \leq k \leq n$ and $1 \leq i \leq k-2$. Denote by the G_i's the matrices of the left braid group actions by the g_i's in $\mathcal{B}'$. Define $H_1 = H_2 = G_1$ and $H_i = G_{i-1}$ for every integer i with $3 \leq i \leq n$. Since $n \geq 5$, our restrictions on r impose in particular $r^8 \neq -1$. Then, the H_i's define a matrix representation for $S^{(0),(n-2,1,1)}$. If $S^{(0),(n-2,1,1)}$ occurs in the Cohen–Wales space, then there exists a basis of $\frac{(n-1)(n-2)}{2}$ vectors

$$W_1^{(3)}, W_1^{(4)}, W_2^{(4)}, W_1^{(5)}, W_2^{(5)}, W_3^{(5)}, \ldots, W_1^{(n)}, \ldots, W_{n-2}^{(n)}$$

of the Cohen–Wales space such that the matrices of the left Cohen–Wales actions by the g_i's, $1 \leq i \leq n$ in this basis are the H_i's, $1 \leq i \leq n$. The left Cohen–Wales action by the g_i's, $1 \leq i \leq n$ on the basis $(\overline{w_{ij}})_{1\leq i<j\leq n}$ of the Cohen–Wales space is the Cohen-Wales representation constructed and defined in Theorem 1 of [9] and whose structure is studied in this paper.

Lemma 2. *The vectors $W_i^{(k)}$'s, $3 \leq k \leq n$, $1 \leq i \leq k-2$, do not contain any hat terms.*

PROOF. First, we show that this is true for the vectors $W_{j-2}^{(j)}$ with $3 \leq j \leq n$. Second, we show that the result also holds for all the other vectors $W_k^{(j)}$ with $3 \leq j \leq n$ and $1 \leq k \leq j-3$. To do so, we proceed by descending induction on the integer k.
We use the second and fourth equalities of Lemma 3.16 of [8] to derive the relations

$$\begin{cases} g_{j-1}.\, W_{j-2}^{(j)} &= -\frac{1}{r}\, W_{j-2}^{(j)} \\ \\ g_j.\, W_{j-2}^{(j)} &= -\frac{1}{r}\, W_{j-2}^{(j)} \end{cases}$$

These two relations imply that $W^{(j)}_{j-2}$ is a linear combination of basis vectors from the Cohen-Wales space, such that their nodes either start or end in $j-1$ or start in $j-2$ and end in j. Further, the first (resp second) relation implies that there is no term in $\widehat{w_{j-2,j-1}}$ (resp $\widehat{w_{j-1,j}}$) in $W^{(j)}_{j-2}$. Furthermore, by the way the coefficients are related as in Lemma 2 of § 3.3 of [9] with $\lambda = -\frac{1}{r}$, we see that $W^{(j)}_{j-2}$ is a multiple of

$$w_{j-2,j-1} - \frac{1}{r}\, w_{j-2,j} + \frac{1}{r^2}\, w_{j-1,j}$$

Fix $j \geq 4$ and suppose the vector $W^{(j)}_k$ does not contain any hat term, where $2 \leq k \leq j-2$. We show this implies that $W^{(j)}_{k-1}$ does not contain any hat term either. From the first equation in Lemma 3.16 of [8], we derive

$$g_k.\, W^{(j)}_k = W^{(j)}_{k-1} + r\, W^{(j)}_k - r^{j-k-1}\, W^{(k+1)}_{k-1} \qquad (\star)$$

By the defining relations of the representation in Theorem 1 of [9], if the vector $W^{(j)}_k$ does not contain any hat terms, since $k \geq 2$, the vector $g_k.\, W^{(j)}_k$ does not contain any hat term either. Moreover, by the first step, $W^{(k+1)}_{k-1}$ does not contain any hat term. Then, with $(\star)$, we conclude that $W^{(j)}_{k-1}$ does not contain any hat term. By induction, the statement of the lemma holds for all the $W^{(j)}_s$'s with $1 \leq s \leq j-2$.
Using Lemma 1, it is now easy to conclude. Indeed, take for instance without loss of generality

$$W^{(3)}_1 = w_{12} - \frac{1}{r}\, w_{13} + \frac{1}{r^2}\, w_{23},$$

and notice an action to the left by g_1 on $W^{(3)}_1$ creates a term in $\widehat{w_{23}}$ with coefficient $-\frac{1}{r}$. Then, by the lemma, $g_1.\, W^{(3)}_1$ cannot be a linear combination of $W^{(k)}_s$'s, which constitutes a contradiction. We conclude that $S^{(0),(n-2,1,1)}$ cannot occur in V_n.

Result 4. *Let $n \geq 5$. The Specht module $S^{(0),(3,1^{n-3})}$ cannot occur in the Cohen-Wales space V_n.*

Proof. Suppose $\mathcal{W}$ is an invariant subspace of V_n that is isomorphic to the Specht module $S^{(0),(3,1^{n-3})}$. Notice when $n \geq 7$, the dimension of $\mathcal{W}$ is greater than $2(n-1)$ and so $\mathcal{W} \cap V_{n-1} \neq \{0\}$. Moreover, by the beginning of § 3.4 of [9], we know that the Specht module $S^{(0),(2,1^{n-3})}$ does not occur in V_{n-1} for any $n \geq 5$. So, whenever $n \geq 7$, we get $\mathcal{W} \cap V_{n-1} \simeq S^{(0),(3,1^{n-4})}$. Thus, by induction, the cases $n \geq 7$ reduce to the case $n = 6$. Let $\mathcal{W} \subset V_6$ such that $\mathcal{W} \simeq S^{(0),(3,1^3)}$. If $\mathcal{W} \cap V_5 \neq \{0\}$, then $\mathcal{W} \cap V_5 \simeq S^{(0),(3,1,1)}$ and so the case $n = 6$ reduces to the case $n = 5$. This case is treated below. We show that the intersection $\mathcal{W} \cap V_5$ is indeed nonzero. An irreducible matrix representation for $S^{(0),(3,1^3)}$ is the one given by the matrices H_i's introduced in the proof of

Result 3, where r has been replaced by $-\frac{1}{r}$ and where $n = 6$. Denote these new matrices by the K_i's. So, there exists a basis $(U_i^{(k)})_{1\leq k\leq 6, 1\leq i\leq k-2}$ of vectors of $\mathcal{W}$ such that the matrices of the left Cohen-Wales actions by the g_i's in this basis are the K_i's. From the last equation of Lemma 3.16 of [8], we derive

$$g_i.\,U_s^{(k)} = -\frac{1}{r}\,U_s^{(k)} \qquad \forall i \notin \{s, s+1, s+2, k\}$$

In particular, we have

$$g_i.\,U_4^{(6)} = -\frac{1}{r}\,U_4^{(6)} \qquad \forall i \leq 3$$

This implies $U_4^{(6)}$ is a linear combination of w_{12}, w_{23} and w_{13}. So, $U_4^{(6)}$ belongs to $\mathcal{W} \cap V_5$.

Thus, it suffices to deal with the case $n = 5$. We must show that it is impossible to have an invariant subspace of V_5 that is isomorphic to the Specht module $S^{(0),(3,1,1)}$. Observe this is Result 3 for $n = 5$.

With Result 3 and Result 4, Proposition 2 and Proposition 3 simplify nicely and become Proposition 5 and Proposition 6 respectively.

Proposition 5. *Assume $(\mathcal{H})_{10}$ holds. Then, for any $n \geq 11$, the Specht modules that may occur in the Cohen-Wales space V_n have dimensions*

$$1, n-1, n, \frac{n(n-3)}{2}, \frac{n(n-1)}{2}, n(n-2)$$

Proposition 6. *Assume Theorem 1 holds for integers n with $4 \leq n \leq 10$. Then, for any $n \geq 4$, the Specht modules that may occur in the Cohen-Wales space V_n are*

$$S^{(0),(n)}, S^{(0),(n-1,1)}, S^{(0),(n-2,2)}, S^{(1),(n-1)}, S^{(1),(n-2,1)}, S^{(2),(n-2)}$$

We now show even more properties on the representation. We prove in particular that point (i) of Theorem 5 of [9] is in fact an equivalence. We have the following result.

Theorem 3. *Let $n \geq 4$ and suppose $l = -\frac{1}{r^{2n-5}}$. Then, $K(n)$ is the unique invariant subspace of V_n and $k(n) = n$. In particular, there exists in V_n a unique irreducible n-dimensional invariant subspace.*

PROOF. With Mathematica, we check that for these values of l and r, we have $k(n) = n$ for every $4 \leq n \leq 7$. Suppose $n \geq 8$. Then, by using points (i) and (iii) of Theorem 8 in [9] on one hand, Theorems 3 and 4 of [9] on the other hand and with our restrictions on r, we see that $k(n) \geq n$. When $n \geq 7$, we have $2n \leq \frac{n(n-3)}{2}$. Next, if $k(n) \geq 2n$, then $k(n) > 2n-2$ and so $K(n) \cap V_{n-1} \neq \{0\}$. This is impossible, because for these values of l and r, the representation $\nu^{(n-1)}$ is irreducible. When $n = 8$, one cannot have $k(8) > 14$ by the same argument. Moreover, if $k(8) = 14$, then $K(8)$ is irreducible and $K(8) \simeq S^{(0),(4,4)}$ or

$K(8) \simeq S^{(0),(2^3)}$. In the first situation, we get $l = r^3$ by the proof of Lemma 3, point (i) of [9]. This contradicts $l = -\frac{1}{r^{11}}$ and $r^{2(8-1)} \neq -1$. The second situation is impossible by the proof of Lemma 3, point (ii) of [9]. Thus, when $l = -\frac{1}{r^{2n-5}}$, we have $k(n) = n$ for all n. Then, $K(n)$ is irreducible, hence is the unique invariant subspace of V_n.

We now show that the statement of Theorem 6 of [9] is in fact an equivalence for $n = 4$ and $n \geq 6$ under $(\mathcal{H})_8$, $(\mathcal{H})_9$ and $(\mathcal{H})_{10}$ (Recall Theorem 6 of [9] also holds for $n = 4$ as shown by Lemma 1 of the current paper).

Theorem 4. *Let $n \geq 4$. Suppose $(\mathcal{H})_8$, $(\mathcal{H})_9$ and $(\mathcal{H})_{10}$ hold. If $l = r^3$, then $k(n) = \frac{n(n-3)}{2}$ and $K(n)$ is the unique invariant subspace of V_n. In particular, there exists in V_n a unique irreducible $\frac{n(n-3)}{2}$-dimensional invariant subspace.*

PROOF. The equality on the dimension is true for $4 \leq n \leq 7$ by using Mathematica. Suppose $n \geq 8$ and suppose $k(n-1) = \frac{(n-1)(n-4)}{2}$. By Theorems $3, 4, 5$ of [9] and Theorem 8, points (i) and (iii) of [9], we have $k(8) \geq 14$ and $k(n) \geq \frac{n(n-3)}{2}$ when $n \geq 9$. Further, under the assumptions of the theorem, by Proposition 1, by the study in § 3.2 of [9], by Results 3 and 4 above and by Theorem 2 of this paper, we see that the irreducible invariant subspaces that may occur in V_n have dimensions $\frac{n(n-3)}{2}$, or $n(n-2)$, except when $n = 8$ when they can also have dimension 35. We first deal with the case $n \geq 9$. Suppose $k(n) \geq n(n-3)$. Then we have

$$k(n-1) \geq \dim(K(n) \cap V_{n-1}) \geq n(n-3) - 2(n-1)$$

Observe the member to the right of the second inequality is greater than $\frac{n(n-4)}{2}$ as soon as $n \geq 6$. This contradicts $k(n-1) = \frac{(n-1)(n-4)}{2}$.
Suppose now $n = 8$. It is shown in forthcoming Result 5 that there does not exist in V_8 any irreducible invariant subspaces of dimension 14. Hence, like in the general case, we get $k(8) \geq 20$ and the dimensions of the irreducible invariant subspaces that may occur in V_8 are the following: 20, 35, 48. Like in the general case, it is impossible to have $k(8) \geq 40$. Hence, suppose $k(8) = 35$. But then

$$k(7) \geq \dim(K(8) \cap V_7) \geq 35 - 2 \times 7 = 21,$$

which contradicts $k(7) = 14$. Thus, we have $k(8) = 20$.
Hence, by induction, $k(n) = \frac{n(n-3)}{2}$ for all $n \geq 4$. Consequently also, the module $K(n)$ is irreducible. Therefore, $K(n)$ is the unique invariant subspace of V_n when $l = r^3$.

The next theorem studies reducibility in the case $l = -r^3$ with some restrictions.

Theorem 5. *Assume $(\mathcal{H})_{10}$ holds. Let n be an integer with $n \geq 11$. Suppose $l = -r^3$ and $r^{2(n-1)} \notin \{i, -i\}$. Then, there exists in V_n an irreducible $n(n-2)$-dimensional invariant subspace.*

PROOF. Immediate by Proposition 5 of the current paper, Theorems 3, 4, 5, 6 of [9] and Theorem 2 of the current paper.

Theorem 6. *Let $n \geq 4$. Suppose $(\mathcal{H})_8$, $(\mathcal{H})_9$ and $(\mathcal{H})_{10}$ hold. If there exists in V_n an irreducible $n(n-2)$-dimensional invariant subspace, then $l = -r^3$.*

PROOF. When $n = 4$, Mathematica gives the value of $k(4)$ depending on the values for l and r. We deduce that if there exists an irreducible 8-dimensional invariant subspace in V_4, the only possibility is to have $l = -r^3$. The case $n = 5$ is also done with Mathematica and is similar. Assume now $n \geq 6$. Since when $n \geq 5$, we have $n(n-2) > 2(2n-3)$, the existence of an irreducible $n(n-2)$-dimensional invariant subspace of V_n implies that $\nu^{(n-1)}$ and $\nu^{(n-2)}$ are both reducible. Then, we must have $l \in \{r^3, -r^3, \frac{1}{r}\}$. Further, by Theorem 7 of [9], when $l = \frac{1}{r}$, there exists an irreducible $\frac{n(n-1)}{2}$-dimensional invariant subspace in V_n. But $n(n-2) + \frac{n(n-1)}{2} > n(n-1)$, so this is impossible. Furthermore, by Theorem 4 above, when $l = r^3$, there exists in V_n an irreducible $\frac{n(n-3)}{2}$-dimensional invariant subspace. This is where the assumptions in the statement of Theorem 6 become relevant. Observe when $n \geq 6$, we have $\frac{n(n-3)}{2} > n$. Thus, we get a contradiction since $n(n-2) + n$ is the degree of the Cohen-Wales representation. We conclude that $l = -r^3$.
We are now ready to conclude when $n \geq 11$, assuming Theorem 1 holds for the small values of n. We have the following statement.

Proposition 7. *Assume Theorem 1 holds for $4 \leq n \leq 10$. Then Theorem 1 holds for every $n \geq 11$.*

PROOF. For point (i), immediate by gathering all the results of [9] and of this paper.
For point (ii), it remains to show that there exists in V_n an irreducible $n(n-2)$-dimensional invariant subspace. By Theorem 10 point (ii) of [9], we know that $K(n) \cap V_{n-1} \neq \{0\}$ since there is an element of V_5 that belongs to $K(n)$ for all $n \geq 5$. Moreover, by point (i) of Theorem 1, we know that $K(n-1)$ is irreducible and that $k(n-1) = (n-1)(n-3)$. Then, $K(n) \cap V_{n-1} = K(n-1)$ and $K(n)$ cannot be one-dimensional. By Proposition 5 and Theorem 2 of the current paper, and by Theorems $3, 4, 5, 6$ of [9], we must have $k(n) = 1 + n(n-2)$. It yields the result.
It remains to show Theorem 1 indeed holds when $4 \leq n \leq 10$. We will show that by excluding a few Specht modules $S^{(0),\lambda}$, we can in fact exclude many more. This is the meaning of Proposition 8 below. First, we need to establish the following result.

Result 5. *In V_8, the Specht modules $S^{(0),(4,4)}$ and $S^{(0),(2^4)}$ don't occur.*

PROOF. The proof is identical to the one of Proposition 3 in § 4 of [9].

Proposition 8. *If $(\mathcal{H})_6$ holds and $Q_n(3) \setminus Q_n(2) = \emptyset$ when $7 \leq n \leq 10$, then $(\mathcal{H})_k$ holds for every $k \geq 4$.*

PROOF. By the same arguments as in the proof of Proposition 1 , it suffices to show that if a module $\mathcal{W}$ belongs to $Q_n(m) \setminus Q_n(3)$ with $7 \leq n \leq 10$ and $m \geq 4$, then $\mathcal{W} \cap V_{n-1} \neq \{0\}$. Applying James'theory in [6], when $n \geq 9$, an irreducible $\mathcal{H}(D_n)$-module $S^{(0),\lambda}$ either belongs to $Q_n(2)$ or has dimension greater than $\frac{(n-1)(n-2)}{2}$. When $n \geq 6$, we have $\frac{(n-1)(n-2)}{2} \geq 2(n-1)$, so this settles $n \in \{9, 10\}$. When $n = 7$, the same statement holds with the exception of $S^{(0),(4,3)}$ and its conjugate Specht module. But both modules belong to $Q_7(3)$, which by assumption is forbidden. Finally, when $n = 8$, the statement above is not true because of the Specht module $S^{(0),(4,4)}$ and its conjugate, both of dimension 14. But we have seen in Result 5 that they cannot occur. So we are done with all the cases.

Lemma 3. *If $S^{(0),(3,2,1)}$ does not occur in V_6, then $(\mathcal{H})_6$ holds.*

PROOF. By the end of § 3.4 in [9], the Specht module $S^{(0),(3,3)}$ and its conjugate $S^{(0),(2^3)}$ cannot occur in V_6.

Using Proposition 8 and Lemma 3, we derive Proposition 9.

Proposition 9. *(Sketch of the final stage of the proof). To finish the proof of Theorem 1, it suffices to show that the following list of Specht modules cannot occur in the Cohen-Wales space V_n. We also provide their dimensions and recall the space in which they may have lived. The integer k denotes the size of the first partition in the double partition.*

$k=0$	$S^{(0),(3,2,1)}$	16	V_6
	$S^{(0),(4,2,1)}$, $S^{(0),(3,2,1^2)}$	35	V_7
	$S^{(0),(6,3)}$, $S^{(0),(2^3,1^3)}$	48	V_9
	$S^{(0),(7,3)}$, $S^{(0),(2^3,1^4)}$	75	V_{10}
	$S^{(0),(4,1^3)}$	20	V_7
	$S^{(0),(5,1^3)}$, $S^{(0),(4,1^4)}$	35	V_8
	$S^{(0),(6,1^3)}$, $S^{(0),(4,1^5)}$	56	V_9
	$S^{(0),(7,1^3)}$, $S^{(0),(4,1^6)}$	84	V_{10}
$k=1$	$S^{(1),(3,3)}$, $S^{(1),(2^3)}$	35	V_7
$k=3$	$S^{(3),(1^3)}$	20	V_6
	$S^{(3,3)^+}$, $S^{(1^3,1^3)^+}$	10	V_6
	$S^{(3,3)^-}$, $S^{(1^3,1^3)^-}$	10	V_6
	$S^{(3),(4)}$, $S^{(1^3),(1^4)}$	35	V_7
	$S^{(3),(1^4)}$, $S^{(1^3),(4)}$	35	V_7
$k=4$	$S^{(4,4)^+}$, $S^{(1^4,1^4)^+}$	35	V_8
	$S^{(4,4)^-}$, $S^{(1^4,1^4)^-}$	35	V_8

PROOF. The dimensions of the modules of $Q_n(3) \setminus Q_n(2)$ are provided in [9] at the end of § 3.2. The dimension $\frac{n(n-2)(n-4)}{3}$ of $S^{(0),(n-3,2,1)}$ and its conjugate

Specht module is too large as soon as $n \geq 8$. Next, we have $\frac{n(n-1)(n-5)}{6} \geq n(n-1)$ if and only if $n \geq 11$. So $S^{(0),(n-3,3)}$ and its conjugate must be considered when $n \in \{9,10\}$. When $n = 7$, it is shown in [9] § 4 that $S^{(0),(4,3)}$ and $S^{(0),(2^3,1)}$ cannot occur. This is Proposition 3 of [9]. When $n = 8$, we saw along the proof of Theorem 2 that $S^{(0),(5,3)}$ and $S^{(0),(2^3,1^2)}$ cannot occur in V_8. As for $S^{(0),(n-3,1^3)}$ and its conjugate, again $\frac{(n-1)(n-2)(n-3)}{6} \geq n(n-1)$ if and only if $n \geq 11$, so these Specht modules must be considered for all the values $7 \leq n \leq 10$. Once we have excluded all these modules from the row $k = 0$, hypothesis $(\mathcal{H})_k$ holds for all $k \geq 4$ by Proposition 8. In particular, $(\mathcal{H})_8$, $(\mathcal{H})_9$ and $(\mathcal{H})_{10}$ hold. So, Theorems $2, 4, 6$ become true for every integer $n \geq 4$ without any assumption. Also, point (i) of Theorem 1 now holds for every integer $n \geq 11$ without any assumption. Further, if we succeed to exclude the modules from the rows $k = 1$, $k = 3$ and $k = 4$, which appear to be the only potential candidates which have not yet been studied by § 3.2 of [9], then Proposition 5 becomes true for every integer $n \geq 4$. Then, we see that Theorem 5 holds for every integer $n \geq 4$ under the only assumption that $r^{2(n-1)} \notin \{i, -i\}$. Then, by gathering all the results, the proof of Theorem 1, point (i) is complete. As for (ii), the proof is the same as in Proposition 7 when $n \geq 6$. And when $n = 4$ (resp $n = 5$), the value of $k(4)$ (resp $k(5)$) obtained with Mathematica forces the existence of an irreducible invariant subspace of V_4 (resp V_5) of dimension 8 (resp 15). The proof is then complete.

Proposition 10. *(End of the proof). The Specht modules from the list of Proposition* 8 *cannot occur in the Cohen-Wales representation.*

PROOF. We first deal with the row $k = 0$. Suppose there exists in V_6 an invariant subspace that is isomorphic to $S^{(0),(3,2,1)}$. Since $16 > 10$, we know that $\mathcal{W} \cap V_5 \neq \{0\}$. By Lemma 3 point (ii) of [9], $S^{(0),(2,2,1)}$ cannot occur in V_5 and by Results 3 or 4, $S^{(0),(3,1,1)}$ cannot occur in V_5, hence $\mathcal{W} \cap V_5 \simeq S^{(0),(3,2)}$. This implies $l = r^3$ by Lemma 3, point (i) of [9]. But when $l = r^3$, we have $k(6) = 9$. Then, it is impossible to have an invariant subspace of dimension 16. So, $S^{(0),(3,2,1)}$ does not occur in V_6 and consequently $(\mathcal{H})_6$ holds. Next, since $20 > 12$, it is an immediate consequence of Results 3 and 4 and the branching rule that $S^{(0),(4,1^3)}$ cannot occur in V_7. If $S^{(0),(4,2,1)}$ or its conjugate occur, then it is impossible to have $l = r^3$, as otherwise, with Mathematica, $k(7) = 14 < 35$. So, if $\mathcal{W} \subset V_7$ is such that $\mathcal{W} \simeq S^{(0),(4,2,1)}$, it follows that $\mathcal{W} \cap V_6 \simeq S^{(0),(4,1,1)}$. This is forbidden by Result 3. Also, if $\mathcal{W} \simeq S^{(0),(3,2,1^2)}$, then $\mathcal{W} \cap V_6 \simeq S^{(0),(3,1^3)}$ or $\mathcal{W} \cap V_6 \simeq S^{(0),(3,2,1)}$, as $S^{(0),(2^2,1^2)}$ cannot occur in V_6 by Lemma 3 point (ii) of [9]. But $S^{(0),(3,1^3)}$ cannot occur in V_6 by Result 4 and we have seen above that $S^{(0),(3,2,1)}$ cannot occur in V_6 either. So, we are done with the second column and we have shown that $(\mathcal{H})_7$ holds at this point. We now deal with the sixth column. Suppose $\mathcal{W} \subset V_8$ with $\mathcal{W} \simeq S^{(0),(5,1^3)}$. Then, by using Result 3, we must have $\mathcal{W} \cap V_7 \simeq S^{(0),(4,1^3)}$. But we have seen above that $S^{(0),(4,1^3)}$ cannot occur in V_7. Similarly, by using Result 4, the Specht module $S^{(0),(4,1^4)}$ cannot occur in V_8. Hence $(\mathcal{H})_8$ holds. Then, by the proof of Theorem 4, if $l = r^3$,

we have $k(8) = 20$. Suppose that $\mathcal{W} \subset V_9$ is isomorphic to one of the Specht modules present in columns 3 and 7. Then, $k(9) \geq 48$ and so

$$\dim(K(9) \cap V_8) \geq 48 - 16 = 32 > 20$$

Thus, it is impossible to have $l = r^3$. We use this fact to rule out the first Specht module present in column 3. For column 7, we conclude directly as follows. If the Specht modules of column 6 cannot occur in V_8, then those of column 7 cannot occur in V_9 by using Results 3 and 4. By the same arguments, the last column vanishes in turn. We now deal with column 3. Since $l \neq r^3$, the Specht module $S^{(0),(6,2)}$ cannot occur in V_8 by Lemma 3 point (i) of [9]. Since $S^{(0),(5,3)}$ can also not occur in V_8 by the proof of Theorem 2, we see that $S^{(0),(6,3)}$ cannot occur in V_9. Next, if $\mathcal{W} \simeq S^{(0),(2^3,1^3)}$, by using the proof of Theorem 4 of [9] and Lemma 3 point (ii) of [9], we see that $\mathcal{W} \cap V_7 \simeq S^{(0),(2^3,1)}$. Further, we have $48 > 42 = 72 - 30$, so that $\mathcal{W} \cap V_6 \neq \{0\}$. By using again Lemma 3 point (ii) of [9], we get $\mathcal{W} \cap V_6 \simeq S^{(0),(2^3)}$. But this is impossible as shown in [9] at the end of § 3.4. At this stage, having successfully ruled out columns 3 and 7, we conclude that $(\mathcal{H})_9$ holds. It remains to deal with column 4. By the fact that $(\mathcal{H})_9$ holds, if $l = r^3$, we have $k(9) = 27$, as in the proof of Theorem 4. If an invariant subspace of dimension 75 occurs in V_{10}, its intersection with V_9 has a dimension greater than or equal to 57 and must be contained in $K(9)$. Therefore, it is impossible to have $l = r^3$. This settles $S^{(0),(7,3)}$ by column 3 and Lemma 3, point (i) of [9]. As for $S^{(0),(2^3,1^4)}$, the conclusion is even more straightforward and follows directly from column 3 and Lemma 3, point (ii) of [9].

We now process the other rows. Using Proposition 8 and our work on the row $k = 0$, we keep in mind that Theorems 4 and 6 now hold, always. Suppose $\mathcal{W} \simeq S^{(1),(3,3)}$. By Theorem 6, the existence of an irreducible invariant subspace of dimension 35 forces $l = -r^3$. Moreover, since $S^{(0),(3,3)}$ cannot occur in V_6 as seen several times in the past, we have $\mathcal{W} \cap V_6 \simeq S^{(1),(3,2)}$. Now the contradiction comes from

$$(\mathcal{W} \cap V_6) \downarrow_{\mathcal{H}(D_5)} \simeq S^{(0),(3,2)} \oplus S^{(1),(2,2)} \oplus S^{(1),(3,1)}$$

Indeed, by the proof of Lemma 3, point (i), the presence of $S^{(0),(3,2)}$ in the restriction module $\mathcal{W} \cap V_6 \downarrow_{\mathcal{H}(D_5)}$ forces $l = r^3$. For $S^{(1),(2^3)}$, the proof is similar. We use the fact that $S^{(0),(2^3)}$ does not occur in V_6 (see the end of § 3.4 of [9]) and the fact that $S^{(0),(2,2,1)}$ cannot be a constituent of $(\mathcal{W} \cap V_6) \downarrow_{\mathcal{H}(D_5)}$ by the proof of Lemma 3, point (ii) of [9]. So, we are done with the row $k = 1$.

We now deal with the first column of the row $k = 3$. Let $\mathcal{W} \subset V_6$ such that $\mathcal{W} \simeq S^{(3),(1^3)}$. Then, by the branching rule,

$$\mathcal{W} \downarrow_{\mathcal{H}(D_4)} \simeq S^{(1),(1^3)} \oplus 2\, S^{(2),(1^2)} \oplus S^{(1),(3)}$$

This is impossible by the end of § 3.5 of [9], where it is proven that $S^{(1),(1^3)}$ cannot occur in any Cohen-Wales space V_n. We now deal with columns 2 and

3. Suppose first $\mathcal{W} \subset V_6$ is such that $\mathcal{W} \simeq S^{(1^3,1^3)^+}$ or $\mathcal{W} \simeq S^{(1^3,1^3)^-}$. We get

$$\mathcal{W} \downarrow_{\mathcal{H}(D_4)} \simeq S^{(1^2,1^2)^+} \oplus S^{(1^2,1^2)^-} \oplus S^{(1),(1^3)}$$

By going back to the proof of Theorem 5 of [9], we see that this is impossible: $S^{(1),(1^3)}$ cannot be a constituent of $\mathcal{W} \downarrow_{\mathcal{H}(D_4)}$. Note this same argument rules out $S^{(1^3),(1^4)}$ in column 4. To show that $S^{(3,3)^+}$ and $S^{(3,3)^-}$ don't occur in V_6, we use computer means. Suppose at least one of them occurs in V_6. Since $k(6) \geq 10$, then by using Mathematica we must have $k(6) = 15$ and $l = \frac{1}{r}$ or $k(6) \geq 24$ and $l = -r^3$. If the first situation occurs, then there must exist an irreducible 5-dimensional invariant subspace in V_6, which also forces $l = \frac{1}{r^5}$ by Theorem 4 of [9]. Hence a contradiction in that case. The dimensions of the irreducible $\mathcal{H}(D_6)$-modules that may occur in V_6 are

$$1, 5, 6, 9, 10, 15, 24$$

Suppose $l = -r^3$. For these values of l and r, there cannot exist any irreducible invariant subspace of dimension 5 (resp 6, resp 9, resp 15), as otherwise $l = \frac{1}{r^5}$ (resp $l = -\frac{1}{r^7}$, resp $l = r^3$, resp $l = \frac{1}{r}$) by Theorem 4 of [9] (resp Theorem 5 of [9], resp Theorem 6 of [9], resp Theorem 2 of the current paper). Nor can there exist an irreducible invariant subspace of V_6 of dimension 24 since $10 + 24 = 34 > 30 = 6 \times 5$. Then it is impossible to have $k(6) \geq 24$. We are now ready to process the rest of the fourth column. Suppose $\mathcal{W} \subset V_7$ is such that $\mathcal{W} \simeq S^{(3),(4)}$. Then we have

$$\mathcal{W} \downarrow_{\mathcal{H}(D_6)} \simeq S^{(3,3)^+} \oplus S^{(3,3)^-} \oplus S^{(2),(4)}$$

On one hand, since $35 > 12$, we get that $\mathcal{W} \cap V_6$ is isomorphic to $S^{(2),(4)}$. Then, the existence of an irreducible 15-dimensional invariant subspace in V_6 forces $l = \frac{1}{r}$ by Theorem 2. On the other hand, the fact that $\mathcal{W}$ has dimension 35 forces $l = -r^3$ by Theorem 6. We obtain a contradiction. So we are done with column 4. Further, it is easy to rule out $S^{(1^3),(4)}$ (resp $S^{(3),(1^4)}$) in column 5 by noticing that $35 > 12$ and by using the first column and Theorem 2 (resp Result 2) with $n = 6$. Finally, by using the fourth column of the row $k = 3$ and the fact that $35 > 14$, we see that none of the Specht modules present in the last row can occur in V_8. This finishes the proof of Theorem 1.

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